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ANALYTIC INVARIANTS AND THE SCHWARZ–PICK INEQUALITY

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ABSTRACT

We find numerical analytic invariants distinguishing between the infinite dimensional analogues of the classical Cartan domains of different type. Further, we define an invariant Hermitian metric on the classical bounded symmetric domains and certain infinite dimensional analogues and show that of all such metrics this is the only one (up to a constant multiple) which yields the best constant in the Schwarz-Pick inequality.

Our purpose is to show that the infinite dimensional analogues of the classical Cartan domains of different types are not holomorphically equivalent and to introduce an invariant Hermitian metric on a class of bounded symmetric domains in Banach spaces (including all the classical bounded symmetric domains) which yields the best constant in the Schwarz–Pick inequality. The domains we consider are the open unit balls of spaces of operators called J^*-algebras. It is shown in [4] that many holomorphic properties of these domains can be expressed in terms of the algebraic properties of the associated J^*-algebra. (See also [9].) For example, two of the domains are holomorphically equivalent if and only if their associated J^*-algebras have the same J^*-structure. Thus domains of different type are not holomorphically equivalent except in dimensions $\leqq 6$ since the maximum dimension m of the space generated by pairs of minimal elements in a J^*-algebra depends only on the J^*-structure and m has different values for domains of different type in dimension > 6. Previous proofs of this result given by E. Cartan [1] and K. H. Look [12] are less simple and do not apply in infinite dimensions.

Our Schwarz–Pick inequalities are proved for the open unit balls of J^*-algebras having finite rank. The rank of a J^*-algebra is the maximum number of

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mutually orthogonal non-zero minimal elements which can be found in the J^*-algebra and thus depends only on the J^*-structure. The rank of a finite dimensional J^*-algebra agrees with the rank of its open unit ball as a Hermitian symmetric space. On each J^*-algebra of finite rank r, we define an inner product in terms of the minimal elements and show that it induces an invariant infinitesimal Hermitian metric on the open unit ball of the J^*-algebra with Schwarz constant $\sqrt{r}$. We also show that any other infinitesimal metric with these properties must be a scalar multiple of ours, contradicting results of K. H. Look and A. Korányi for the Bergman metric. Further, we give necessary and sufficient conditions for each of Korányi's inequalities to hold for a classical bounded symmetric domain and we obtain an expression for the integrated form of any invariant infinitesimal Hermitian metric on such a domain.

Applications of infinite dimensional bounded symmetric domains to mathematical physics are given in [20].

1. Preliminary definitions and notation

Let H and K be complex Hilbert spaces and let $\mathcal{L}(H,K)$ denote the Banach space of all bounded linear transformations from H to K with the operator norm. A *J^*-algebra* is a closed complex subspace $\mathfrak{A}$ of $\mathcal{L}(H,K)$ such that $AA^*A \in \mathfrak{A}$ whenever $A \in \mathfrak{A}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are J^*-algebras, a linear map $L: \mathfrak{A} \to \mathfrak{B}$ is said to be a *J^*-isomorphism* if L is a bounded bijection of $\mathfrak{A}$ onto $\mathfrak{B}$ satisfying $L(AA^*A) = L(A)L(A)^*L(A)$ for all $A \in \mathfrak{A}$. (Throughout, unless otherwise indicated, the symbols $\mathfrak{A}$ and $\mathfrak{B}$ denote arbitrary J^*-algebras.)

For example, the sets $\mathcal{L}(H,K)$, $\{A \in \mathcal{L}(H): A^t = A\}$ and $\{A \in \mathcal{L}(H): A^t = -A\}$, where $x \to \bar{x}$ is a given conjugation on H and $A^t = \overline{A^*\bar{x}}$ for all $x \in H$, are J^*-algebras and are called *Cartan factors type* I, II, and III, respectively. Also, any closed complex subspace $\mathfrak{A}$ of $\mathcal{L}(H)$ such that both $A^* \in \mathfrak{A}$ and $A^2 \in \mathbf{C}I$ whenever $A \in \mathfrak{A}$, is a J^*-algebra and is called a *Cartan factor of type* IV except when $\dim \mathfrak{A} = 2$. The Cartan factor of type I, II or III with $\dim H = n$ and $\dim K = m$ is denoted by $\mathrm{I}(m,n)$, $II(n)$ or $\mathrm{III}(n)$, respectively, and the n-dimensional Cartan factor of type IV is denoted by $\mathrm{IV}(n)$.

We say that an operator $B \in \mathfrak{A}$ is a *minimal element of* $\mathfrak{A}$ if for each $A \in \mathfrak{A}$ there is a $\lambda \in \mathbf{C}$ with $BA^*B = \lambda B$. We call operators $A, B \in \mathfrak{A}$ *orthogonal* if both $AB^* = 0$ and $B^*A = 0$. Define an operator $\langle A, B\rangle \in (\mathfrak{A})$ by $\langle A, B\rangle C = \frac{1}{2}(AB^*C + CB^*A)$. Then A and B are orthogonal if and only if $\langle A, B\rangle = 0$ by [4, p. 18].

The open unit ball of $\mathfrak{A}$ is denoted throughout by $\mathfrak{A}_0$. Note that the open unit balls of the finite dimensional Cartan factors of types I–IV are just the Cartan domains of the corresponding types. For each $B \in \mathfrak{A}_0$, the transformation T_B defined by

$$T_B(A) = (I - BB^*)^{-\frac{1}{2}}(A + B)(I + B^*A)^{-1}(1 - B^*B)^{\frac{1}{2}}$$

is a biholomorphic mapping of $\mathfrak{A}_0$ onto itself with $T_B(0) = B$, and

$$DT_B^{-1}(B)A = (I - BB^*)^{-\frac{1}{2}}A(I - B^*B)^{-\frac{1}{2}}$$

for all $A \in \mathfrak{A}$.

2. Minimal elements and holomorphic equivalence

Proposition 1. *The set of minimal elements of a Cartan factor* $\mathfrak{A}$ *of*
type I *is* $\{yx^*: x \in H, y \in K\}$,
type II *is* $\{x\bar{x}^*: x \in H\}$,
type III *is* $\{xy^* - \bar{y}\bar{x}^*: x, y \in H\}$,
type IV *is* $\{B \in \mathfrak{A}: B^2 = 0\}$ *if* $\dim \mathfrak{A} > 1$,
where H and K are the underlying Hilbert spaces for $\mathfrak{A}$.

Proof. It is easy to verify that each of the sets given is a set of minimal elements of the mentioned Cartan factor. (See, for example, the identities in the proof of Lemma 2 below.) Conversely, suppose B is a non-zero minimal element of $\mathfrak{A}$. If $\mathfrak{A}$ is type I–III, there is an $x \in H$ and a $y \in K$ with $\|y\| = 1$ and $y = Bx$. If $\mathfrak{A}$ is of type I, then $\lambda B = B(yx^*)^*B = y(B^*y)^*$ and evaluating at x, we see that $\lambda = 1$. If $\mathfrak{A}$ is of type II, then $\lambda B = B(\bar{x}x^*)^*B = y\bar{y}^*$ since $B' = B$, and evaluating at $\bar{y}$, we see that $\lambda \neq 0$. If $\mathfrak{A}$ is of type III, then since $B' = -B$, we have

$$B\bar{y}\bar{x}^*B = (-\overline{B^*y})(B^*\bar{x})^* = (\overline{B^*y})\bar{y}^*$$

and

$$\bar{x}^*y = (B^*\bar{x})^*x = -\bar{y}^*x = -\bar{x}^*y,$$

which implies $\bar{y}^*x = 0$. Hence

$$\lambda B = B(yx^* - \bar{x}\bar{y}^*)^*B = y(B^*y)^* - (\overline{B^*y})\bar{y}^*$$

and evaluating at x, we see that $\lambda = 1$. If $\mathfrak{A}$ is of type IV and $B^2 \neq 0$, the identity

$$BA^*B = B(A^*B + BA^*) - B^2A^* = 2(B, A)B - (B, B^*)A^*$$

shows that $\mathfrak{A}^* \subseteq \mathbf{C}B$, so $\dim \mathfrak{A} = 1$.

Define a *generalized Cartan factor of type* I–IV to be a J^*-algebra contained in a Cartan factor $\mathfrak{A}$ of the corresponding type and containing all the minimal elements of $\mathfrak{A}$. For example, if $\mathfrak{A}$ is a Cartan factor of type I–III, then the J^*-algebra of all compact operators in $\mathfrak{A}$ is a generalized Cartan factor of the same type. It follows from Proposition 4 (below) that the generalized Cartan factors coincide with the Cartan factors in finite dimensions and that the generalized Cartan factors of type IV always coincide with the Cartan factors of type IV.

Given a J^*-algebra $\mathfrak{A}$, define $m(\mathfrak{A})$ to be the supremum of the dimensions of the spaces $\{B_1A^*B_2 + B_2A^*B_1 : A \in \mathfrak{A}\}$, where B_1 and B_2 vary over all minimal elements of $\mathfrak{A}$. (We allow $m(\mathfrak{A}) = \infty$). Clearly $m(\mathfrak{A}) = m(\mathfrak{B})$ whenever $\mathfrak{A}$ and $\mathfrak{B}$ are J^*-isomorphic J^*-algebras.

LEMMA 2. *If* $\mathfrak{A}$ *is a generalized Cartan factor of*
type I, *then* $m(\mathfrak{A}) = 2$ *unless* $\dim \mathfrak{A} = 1$,
type II, *then* $m(\mathfrak{A}) = 1$,
type III, *then* $m(\mathfrak{A}) = 4$ *unless* $\dim \mathfrak{A} < 4$,
type IV, *then* $m(\mathfrak{A}) = \dim \mathfrak{A} - 2$ *unless* $\dim \mathfrak{A} \leqq 2$.

PROOF. If $\mathfrak{A}$ is of type I and $B_1 = y_1x_1^*$, $B_2 = y_2x_2^*$, where $x_1, x_2 \in H$ and $y_1, y_2 \in K$, then

$$B_1A^*B_2 + B_2A^*B_1 = (y_2, Ax_1)y_1x_2^* + (y_1, Ax_2)y_2x_1^*$$

for all $A \in \mathfrak{A}$. Hence taking $A = y_2x_1^*$ and $A = y_1x_2^*$, we see that $m(\mathfrak{A}) = 2$ unless $\dim \mathfrak{A} = 1$. If $\mathfrak{A}$ is of type II and $B_1 = x_1\bar{x}_1^*$, $B_2 = x_2\bar{x}_2^*$, where x_1, $x_2 \in H$, then

$$B_1A^*B_2 + B_2A^*B_1 = (x_2, A\bar{x}_1)(x_1\bar{x}_2^* + x_2\bar{x}_1^*)$$

for all $A \in \mathfrak{A}$ since $A' = A$, so $m(\mathfrak{A}) = 1$. If $\mathfrak{A}$ is of type III, define $[x, y] = xy^* - \bar{y}\bar{x}^*$ for $x, y \in H$. If $A \in \mathfrak{A}$ and $x, y, z, w \in H$, then

$$[x, y]A^*[z, w] + [z, w]A^*[x, y]$$

$$= (z, Ay)[x, w] - (z, A\bar{x})[\bar{y}, w] + (x, Aw)[z, y] - (\bar{w}, Ay)[x, \bar{z}]$$

since $(\bar{y}, Ax) = -(\bar{x}, Ay)$ for all $x, y \in H$. If $\dim H \geqq 4$, there is an orthonormal set $\{x, y, z, w\}$ of self-conjugate elements of H, and taking $A = [x, w]$, $[y, w]$, $[z, y]$, $[x, z]$ in succession, we see that each of these operators lies in the space in question. Hence $m(\mathfrak{A}) = 4$.

If $\mathfrak{A}$ is a Cartan factor of type IV and B_1, B_2 are minimal elements of $\mathfrak{A}$, then

$$\begin{aligned} B_1A^*B_2 + B_2A^*B_1 &= B_1(A^*B_2 + B_2A^*) - (B_1B_2 + B_2B_1)A^* \\ &\quad + B_2(B_1A^* + A^*B_1) \\ &= -2L(A^*), \end{aligned}$$

where $L(A) = (B_1, B_2^*)A - (A, B_2^*)B_1 - (A, B_1^*)B_2$. Moreover, $L(B_1) = L(B_2) = 0$, so letting $\mathcal{R}$ be the range of L, we have $\dim \mathcal{R} \leqq \dim \mathfrak{A} - 2$ if $\dim \mathfrak{A} > 2$. Let B be a minimal element of $\mathfrak{A}$ with $(B, B) = 1$. Then taking $B_1 = B$ and $B_2 = B^*$, we see that $S = \{B_1, B_2\}$ is an orthonormal set in $\mathfrak{A}$ and that L is the projection of $\mathfrak{A}$ onto the orthogonal complement of S, so $\dim \mathcal{R} = \dim \mathfrak{A} - 2$. Hence, $m(\mathfrak{A}) = \dim \mathfrak{A} - 2$.

THEOREM 3. *The open unit balls of two generalized Cartan factors of different type are holomorphically equivalent if and only if both the generalized Cartan factors are one dimensional or the generalized Cartan factors are one of the J^*-isomorphic pairs* {I(1,3), III(3)}, {II(2), IV(3)}, {I(2,2), IV(4)}, *or* {III(4), IV(6)}.

PROOF. Let $\mathfrak{A}$ and $\mathfrak{B}$ be generalized Cartan factors of different type and suppose $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are holomorphically equivalent. Then by [4, cor. 3], $m(\mathfrak{A}) = m(\mathfrak{B})$ and $\dim \mathfrak{A} = \dim \mathfrak{B}$. Hence by the previous lemma, $\mathfrak{A}$ and $\mathfrak{B}$ are either both one dimensional or one of the pairs {II(2), III(3)}, {II(2), IV(3)}, {III(3), IV(3)}, {I(1,3), III(3)}, {I(2,2), IV(4)}, {I(1,4), IV(4)}, or {III(4), IV(6)}.

Let $\{U_k\}_1^4$ be a self-adjoint orthonormal basis for IV(4). Then

$$z_1U_1 + \cdots + z_4U_4 \to \begin{bmatrix} z_1 + iz_2 & z_3 + iz_4 \\ z_3 - iz_4 & -z_1 + iz_2 \end{bmatrix}$$

is a J^*-isomorphism of IV(4) onto I(2,2), and the restriction of this map to the first three coordinates gives a J^*-isomorphism of IV(3) onto II(2). Let $\{U_k\}_1^6$ be a self-adjoint orthonormal basis for IV(6). Then

$$z_1U_1 + \cdots + z_6U_6 \to \begin{bmatrix} 0 & z_1 + iz_2 & z_3 + iz_4 & z_5 + iz_6 \\ -z_1 - iz_2 & 0 & z_5 - iz_6 & -z_3 + iz_4 \\ -z_3 - iz_4 & -z_5 + iz_6 & 0 & z_1 - iz_2 \\ -z_5 - iz_6 & z_3 - iz_4 & -z_1 + iz_2 & 0 \end{bmatrix}$$

is a J^*-isomorphism of IV(6) onto III(4). (To verify this, first observe that the above matrix is unitary when $z_1, \cdots, z_6$ are real and $z_1^2 + \cdots + z_6^2 = 1$.) Also

$$[z_1, z_2, z_3] \to \begin{bmatrix} 0 & z_1 & z_2 \\ -z_1 & 0 & z_3 \\ -z_2 & -z_3 & 0 \end{bmatrix}$$

is a J^*-isomorphism of I(1,3) onto III(3). Note that {I(1,4), IV(4)} and {I(1,3), IV(3)} are not J^*-isomorphic pairs since not every element of a Cartan factor of type IV of dimension >1 is minimal. Thus pairs 1, 3 and 6 of our list are not J^*-isomorphic pairs.

The holomorphic equivalence of the open unit balls of III(4) and IV(6) was discovered by Morita [17] and Look [12]. It was overlooked by E. Cartan [1, p. 152].

3. Finite rank J^*-algebras

We say that a J^*-algebra $\mathfrak{A}$ has *finite rank* if there exists a number n such that $\sigma(A^*A)$ has at most n non-zero elements for each $A \in \mathfrak{A}$. The least such n is called the *rank* of $\mathfrak{A}$ and is denoted by $r(\mathfrak{A})$.

For example, every finite dimensional J^*-algebra $\mathfrak{A}$ has finite rank and $r(\mathfrak{A}) \leqq \dim \mathfrak{A}$. Indeed, given $n = \dim \mathfrak{A}$ and $A \in \mathfrak{A}$, the operators $A, AA^*A, \cdots, A(A^*A)^n$ are dependent so there is a polynomial $p \not\equiv 0$ of degree $\leqq n$ with $A^*Ap(A^*A) = 0$, and consequently $\sigma(A^*A)$ contains at most n non-zero elements by the spectral mapping theorem. Clearly if $\mathfrak{A} = \mathscr{L}(\mathbf{C}^n, H)$ with $\dim H \geqq n$ or if $\mathfrak{A} = \mathrm{II}(n)$, then $r(\mathfrak{A}) = n$. Also, if $\mathfrak{A}$ is a Cartan factor of type IV, then $r(\mathfrak{A}) = 2$ when $\dim \mathfrak{A} > 1$ by [4, (9)]. Thus many infinite dimensional J^*-algebras have finite rank. We shall see shortly that $r(\mathfrak{A}) = [n/2]$ when $\mathfrak{A} = \mathrm{III}(n)$.

Our next result is a form of the spectral theorem which, in view of Proposition 1, contains the normal form for rectangular, symmetric and skew-symmetric matrices. (Compare [7], [8], and [15].)

PROPOSITION 4. *If $\mathfrak{A}$ has rank n, then for each non-zero $A \in \mathfrak{A}$, there exist mutually orthogonal non-zero minimal partial isometries $V_1, \cdots, V_m$ in $\mathfrak{A}$ and positive numbers $a_1, \cdots, a_m$ such that*

$$A = \sum_{k=1}^{m} a_k V_k \tag{1}$$

and $m \leqq n$. *In fact, when* (1) *holds,* $a_1, \cdots, a_m$ *are the non-zero eigenvalues of* $(A^*A)^{\frac{1}{2}}$ *with possible repetitions.*

We call the numbers $a_1, \cdots, a_m$ the *singular values* of A. (Note that the multiplicities depend on the choice of $\mathfrak{A}$.)

COROLLARY 5. *The rank of* $\mathfrak{A}$, *when finite, is the maximum number of mutually orthogonal non-zero minimal partial isometries in* $\mathfrak{A}$.

COROLLARY 6. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* J^**-isomorphic, then* $\mathrm{r}(\mathfrak{A}) = \mathrm{r}(\mathfrak{B})$. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* J^**-subalgebras of a* J^**-algebra such that each element of* $\mathfrak{A}$ *is orthogonal to each element of* $\mathfrak{B}$, *then* $\mathrm{r}(\mathfrak{A} + \mathfrak{B}) = \mathrm{r}(\mathfrak{A}) + \mathrm{r}(\mathfrak{B})$.

To deduce the rank of $\mathfrak{A} = \mathrm{III}(n)$, observe that the range of a non-zero minimal element of $\mathfrak{A}$ is two dimensional and that the ranges of two orthogonal minimal elements of $\mathfrak{A}$ are orthogonal. Hence given $A \in \mathfrak{A}$, by (1), $2m \leqq n$ and $\sigma(A^*A)$ has at most m non-zero elements, so $\mathrm{r}(\mathfrak{A}) \leqq [n/2]$. That equality holds can be seen by considering a skew-symmetric $n \times n$ matrix whose non-zero entries are located along the alternate diagonal.

PROOF OF PROPOSITION 4 AND ITS COROLLARIES. Let $A \in \mathfrak{A}$ and put $P = (A^*A)^{\frac{1}{2}}$. Then there is a partial isometry W with $A = WP$ and $E = W^*W$ is the projection onto the closure of the range of P. (See [3, p. 68].) By hypothesis and the spectral theorem, there exist mutually orthogonal non-zero projections $E_1, \cdots, E_l$ such that $P = \Sigma_1^l \lambda_j E_j$, where $\lambda_1, \cdots, \lambda_l$ are the non-zero eigenvalues of P. Moreover, for each j, there exists a polynomial q with $E_j = Pq(P^2)$. Clearly, $EE_j = E_jE = E_j$, and putting $W_j = WE_j$, we have $W_j = Aq(A^*A) \in \mathfrak{A}$. Hence $W_1, \cdots, W_l$ are mutually orthogonal non-zero partial isometries in $\mathfrak{A}$ and $A = \Sigma_1^l \lambda_j W_j$.

Let $V_1, \cdots, V_m$ be mutually orthogonal non-zero partial isometries in $\mathfrak{A}$. Then putting $B = \Sigma_1^m kV_k$, we have $B^*B = \Sigma_1^m k^2 V_k^* V_k$ so $\sigma(B^*B)$ has m non-zero elements and thus $m \leqq n$. Hence given j, there is a maximum number of mutually orthogonal non-zero partial isometries in $\mathfrak{A}$ whose sum is W_j. Let V be one of these. Obviously, V is not a sum of two mutually orthogonal non-zero partial isometries in $\mathfrak{A}$. To prove (1), it suffices to show that V is a minimal element of $\mathfrak{A}$. Put $\mathfrak{B} = V^*\mathfrak{A}V^*V$ and let $F = V^*V$. Then $F \in \mathfrak{B}$, $\mathfrak{B}$ contains the adjoint and all powers of each of its elements, and $\mathrm{r}(\mathfrak{B}) \leqq \mathrm{r}(\mathfrak{A})$. Moreover, it is easily verified that F is not a sum of two mutually orthogonal non-zero projections in $\mathfrak{B}$. Let $C \in \mathfrak{B}$ with $C^* = C$. Then by the spectral theorem, there exist non-zero real numbers $c_1, \cdots, c_k$ and mutually orthogonal projections

$F_1, \cdots, F_k$ in $\mathfrak{B}$ such that $C = \Sigma_1^k c_j F_j$. For each j, $F_j F = F F_j = F_j$, which implies that F_j and $F - F_j$ are mutually orthogonal projections in $\mathscr{B}$ and thereofre $F_j = 0$ or $F_j = F$. Thus $C \in CF$, and since $\mathfrak{B}$ is adjoint closed, this relation holds for all $C \in \mathfrak{B}$. It is easy to see that this implies that V is a minimal element of $\mathfrak{A}$.

Suppose (1) holds. Then $A^*A = \Sigma_1^m a_k^2 V_k^* V_k$ and $P = \Sigma_1^m a_k V_k^* V_k$, so $a_1^2, \cdots, a_m^2$ are the non-zero elements of $\sigma(A^*A)$ and $a_1, \cdots, a_m$ are the non-zero eigenvalues of P. Hence if r is the maximum number of mutually orthogonal non-zero minimal partial isometries in $\mathfrak{A}$, then $n \leqq r$. Thus Corollary 5 holds since $r \leqq n$ by what we have already shown. Corollary 6 follows from Corollary 5 since a minimal element of $\mathfrak{A} + \mathfrak{B}$ lies in either $\mathfrak{A}$ or $\mathfrak{B}$.

We remark that it follows from Proposition 4 and [18, p. 85] that a C^*-algebra $\mathfrak{A}$ has finite rank if and only if $\mathfrak{A}$ is finite dimensional. Indeed, if A is given by (1) and if B is given by the same expression with the a_k's replaced by their reciprocals, then $AB^*A = A$, so $\mathfrak{A}$ is von Neumann regular when $r(\mathfrak{A}) < \infty$.

Proposition 7. *The following are equivalent*:

(i) $\mathfrak{A}$ *is J^*-isomorphic to a Hilbert space.*

(ii) *Each operator in* $\mathfrak{A}$ *is a scalar multiple of a partial isometry.*

(iii) $\mathfrak{A}$ *has rank* 1.

(iv) *Each operator in* $\mathfrak{A}$ *of norm* 1 *is a complex extreme point of the closed unit ball of* $\mathfrak{A}$.

Proof. Clearly (i) $\Rightarrow$ (iv), and (iv) $\Rightarrow$ (ii) by [4, theorem 11]. Also, (ii) $\Leftrightarrow$ (iii) by the spectral theorem and [3, p. 63]. Hence it suffices to show that (ii) $\Rightarrow$ (i), and we may assume that $\dim \mathfrak{A} \geqq 2$. Let A and B be linearly independent operators in $\mathfrak{A}$ and put $f(\lambda) = (A + \lambda B)(A + \lambda B)^*(A + \lambda B)$ for $\lambda \in \mathbf{C}$. Assuming (ii) holds, there is a function $\varphi: \mathbf{C} \to \mathbf{C}$ satisfying

$$f(\lambda) = \varphi(\lambda)(A + \lambda B) \tag{2}$$

for all $\lambda \in \mathbf{C}$. It is easy to verify that there exist operators $C_0, \cdots, C_5$ which can be expressed as linear combinations of the values of f such that

$$f(\lambda) = C_0 + \lambda C_1 + \bar{\lambda} C_2 + \lambda^2 C_3 + |\lambda|^2 C_4 + \lambda |\lambda|^2 C_5$$

for all $\lambda \in \mathbf{C}$. Hence there exist numbers $a_0, \cdots, a_5$ and $b_0, \cdots, b_5$ with $C_k = a_k A + b_k B$ for $k = 0, \cdots, 5$, and thus equating the coefficients of A and B in (2), we obtain

$$\varphi(\lambda) = a_0 + a_1\lambda + a_2\bar{\lambda} + a_3\lambda^2 + a_4|\lambda|^2 + a_5\lambda|\lambda|^2,$$

$$\lambda\varphi(\lambda) = b_0 + b_1\lambda + b_2\bar{\lambda} + b_3\lambda^2 + b_4|\lambda|^2 + b_5\lambda|\lambda|^2$$

for all $\lambda \in \mathbf{C}$. Clearly $b_0 = 0$ and φ is continuous at $\lambda = 0$. Hence dividing the last equality by λ and letting λ approach 0 through real and imaginary values, we obtain $b_2 = 0$. Since $C_2 = AB^*A$, this proves that for each $A, B \in \mathfrak{A}$ there is a complex number (A, B) with

$$AB^*A = (A, B)A. \tag{3}$$

If $A = 0$, we define $(A, B) = 0$. Obviously $(A, B) = 0$ when $B = 0$. It is easy to verify that (A, B) is well defined and conjugate-linear in B. Also,

$$(A, B)B^*A = B^*(AB^*A) = (BA^*B)^*A = \overline{(B, A)}B^*A,$$

and if $B^*A = 0$, then both $AB^*A = 0$ and $(BA^*B)^* = 0$, so $(A, B)A = 0$ and $\overline{(B, A)}B^* = 0$. Hence $(A, B) = \overline{(B, A)}$ for all $A, B \in \mathfrak{A}$. Moreover, $(A, A) \geqq 0$ and $(A, A) = \|A\|^2$ since $(A^*A)^2 = (A, A)A^*A$. Thus $\mathfrak{A}$ is a Hilbert space and (3) shows that the given J^*-structure of $\mathfrak{A}$ is identical with the Hilbert J^*-structure of $\mathfrak{A}$.

4. The algebraic inner product

Proposition 8. *If $\mathfrak{A}$ has finite rank, there exists an inner product $(\cdot, \cdot)$ on $\mathfrak{A}$ such that*

(i) $(A, A) = \Sigma_1^m a_k^2$ *for all $A \in \mathfrak{A}$, where $a_1, \cdots, a_m$ are the singular values of A,*

(ii) $\|A\|^2 \leqq (A, A) \leqq \mathrm{r}(\mathfrak{A})\|A\|^2$ *for all $A \in \mathfrak{A}$,*

(iii) $(L(A), L(B)) = (A, B)$ *for all $A, B \in \mathfrak{A}$, whenever $L: \mathfrak{A} \to \mathfrak{A}$ is a J^*-isomorphism,*

(iv) $(\langle A, B\rangle C, D) = (C, \langle B, A\rangle D)$ *for all $A, B, C, D \in \mathfrak{A}$,*

(v) $(A, B) = 0$ *whenever A and B are orthogonal elements of $\mathfrak{A}$.*

We call the inner product of Proposition 8 the *algebraic inner product* for $\mathfrak{A}$.

Proof. For each minimal partial isometry V in $\mathfrak{A}$, define $\ell_V \in \mathfrak{A}^*$ by $VB^*V = \overline{\ell_V(B)}V$. Note that $\ell_V(W) = \overline{\ell_W(V)}$ when V and W are minimal elements of $\mathfrak{A}$ since $(WV^*W)^*V = W^*(VW^*V)$. Given $A \in \mathfrak{A}$, let $a_1, \cdots, a_m$ and $V_1, \cdots, V_m$ be as in (1) and define

$$(A, B) = \sum_{k=1}^{m} a_k\overline{\ell_{V_k}(B)}$$

for $B \in \mathfrak{A}$. Given $B \in \mathfrak{A}$, to see that (A, B) is well defined, let $B = \Sigma_1^\ell b_k W_k$ be a decomposition of B similar to that of A, and observe that

$$\sum_{k=1}^{m} a_k \overline{\ell_{V_k}(B)} = \sum_{k=1}^{m} a_k \sum_{j=1}^{\ell} b_j \overline{\ell_{V_k}(W_j)} = \sum_{j=1}^{\ell} b_j \sum_{k=1}^{m} a_k \ell_{W_j}(V_k) = \sum_{j=1}^{\ell} b_j \ell_{W_j}(A).$$

Thus (A, B) does not depend on the decomposition of A used and $\overline{(A, B)} = (B, A)$. Obviously (i) holds and thus $(\cdot, \cdot)$ is an inner product on $\mathfrak{A}$. Clearly (ii) follows from (i), and (iii) follows from (i) and the fact that J^*-isomorphisms take orthogonal elements to orthogonal elements and minimal elements to minimal elements. To prove (v), let A and B be orthogonal with the decompositions given above. Then

$$0 = W_j^* W_j B^* A V_k^* V_k = a_k b_j W_j^* V_k,$$

$$0 = V_k V_k^* A B^* W_j W_j^* = a_k b_j V_k W_j^*,$$

so V_k and W_j are orthogonal for all k and j. Hence by (i), $(A + \lambda B, A + \lambda B) = (A, A) + (B, B)$ whenever $|\lambda| = 1$, so $(A, B) = 0$.

Finally, to prove (iv), let $A \in \mathfrak{A}$ and $t \in \mathbf{R}$, and define $L_t = \exp(2it\langle A, A\rangle)$. Clearly, $L_t \in \mathcal{L}(\mathfrak{A})$ since $\langle A, A\rangle \in \mathcal{L}(\mathfrak{A})$, and $L_t(C) = \exp(itAA^*)C\exp(itA^*A)$ for all $C \in \mathfrak{A}$ since left multiplication by AA^* commutes with right multiplication by A^*A. Hence L_t is a J^*-isomorphism of $\mathfrak{A}$ onto itself. Given $C, D \in \mathfrak{A}$, $(L_tC, L_tD) = (C, D)$ by (iii), and differentiating at $t = 0$, we have $(\langle A, A\rangle C, D) = (C, \langle A, A\rangle D)$. Thus we obtain (iv) by substituting $A + B$ for A and applying the conjugate linearity of $\langle A, B\rangle$ in B. (Note that our argument shows that (iii) implies (iv) for any sesquilinear form on $\mathfrak{A}$.)

For example, let $\mathfrak{A}$ be a Cartan factor with $\mathrm{r}(\mathfrak{A}) < \infty$. Then $(A, B) = \mathrm{tr} B^*A$ when $\mathfrak{A}$ is of type I or II, $(A, B) = \frac{1}{2}\mathrm{tr} B^*A$ when $\mathfrak{A}$ is of type III, and $(A, B)I = AB^* + B^*A$ when $\mathfrak{A}$ is of type IV. The type I and II cases follow directly from the polarization formula, Proposition 4 and the fact that projections of rank 1 have trace 1. If $\mathfrak{A}$ is of type III and $V = xy^* - \bar{y}\bar{x}^*$ is a non-zero partial isometry in $\mathfrak{A}$, then $VV^*V = \lambda V$, where $\lambda = \|x\|^2\|y\|^2 - |(x, \bar{y})|^2$, so $\mathrm{tr}\, V^*V = 2\lambda = 2\|V\|^2 = 2$. Hence the type III case follows from Proposition 4. If $\mathfrak{A}$ is of type IV, then there is an inner product $(\cdot, \cdot)'$ on $\mathfrak{A}$ such that $2(A, B)'I = AB^* + B^*A$ by [4]. Hence if V is a minimal element of $\mathfrak{A}$, $VB^*V = (VB^* + B^*V)V = 2(V, B)'V$, so $(A, B) = 2(A, B)'$ by definition.

The algebraic inner product appears to be a new construction. (Compare [19, p. 261], [10, p. 99] and [22].) As an application, we obtain the following characterization of rank:

PROPOSITION 9. *Suppose* $\dim \mathfrak{A} < \infty$. *Then* $\mathrm{r}(\mathfrak{A}) = n$ *if and only if there exists maximal set of* n *mutually orthogonal non-zero minimal elements of* $\mathfrak{A}$.

PROOF. Let $\mathscr{E}$ be the set of extreme points of the closed unit ball of $\mathfrak{A}$ and suppose $V_1, \cdots, V_m$ is a maximal set of mutually orthogonal non-zero minimal partial isometries in $\mathfrak{A}$. Put $V = V_1 + \cdots + V_m$. Then the J^*-algebra $(I - VV^*)\mathfrak{A}(I - V^*V)$ has no non-zero minimal partial isometries so $V \in \mathscr{E}$ by Proposition 4 and [4, theorem 11]. Also, $(V, V) = m$ by part (i) of Proposition 8. Now the map $W \to (W, W)$ is continuous on $\mathscr{E}$ and $\mathscr{E}$ is connected by [4, corol. 9], so $(W, W) = m$ for all $W \in \mathscr{E}$. Thus Proposition 9 follows from Corollary 5.

We define a *J^*-ideal* in $\mathfrak{A}$ to be a closed subspace $\mathfrak{J}$ of $\mathfrak{A}$ such that if $A, B, C \in \mathfrak{A}$, then $AB^*C + CB^*A \in \mathfrak{J}$ whenever $B \in \mathfrak{J}$ or $C \in \mathfrak{J}$. For example, the J^*-ideals of a C^*-algebra $\mathfrak{A}$ are precisely the closed ideals of $\mathfrak{A}$ by theorems 4.8.14 and 4.9.2 of [19]. We say that a J^*-ideal is *simple* if the only J^*-ideals in $\mathfrak{J}$ are $\{0\}$ and $\mathfrak{J}$.

THEOREM 10. *Let* $\mathfrak{A}$ *have finite rank. Then there exists a unique set of mutually orthogonal non-zero simple* J^**-ideals* $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$ *in* $\mathfrak{A}$ *such that*

$$\mathfrak{A} = \mathfrak{J}_1 + \cdots + \mathfrak{J}_n. \tag{4}$$

Moreover, if $f: \mathfrak{A} \times \mathfrak{A} \to \mathbf{C}$ *is a continuous sesquilinear form on* $\mathfrak{A}$ *satisfying*

$$f(\langle A, B\rangle C, D) = f(C, \langle B, A\rangle D) \tag{5}$$

for all $A, B, C, D \in \mathfrak{A}$, *then there exist complex numbers* $c_1, \cdots, c_n$ *such that*

$$f(A, B) = \sum_{k=1}^{n} c_k (A_k, B_k)_k \tag{6}$$

for all $A, B \in \mathfrak{A}$, *where* $(\cdot, \cdot)_k$ *is the algebraic inner product for* $\mathfrak{J}_k$ *and where* $A = A_1 + \cdots + A_n$ *and* $B = B_1 + \cdots + B_n$ *are decompositions given by* (4).

COROLLARY 11. *If* $\dim \mathfrak{A} < \infty$, *then each of the* J^**-ideals* $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$ *above is* J^**-isomorphic to one of the Cartan factors of type* I–IV.

PROOF. Clearly the set

$$\Gamma = \{L \in \mathscr{L}(\mathfrak{A}): L\langle A, B\rangle = \langle A, B\rangle L \text{ for all } A, B \in \mathfrak{A}\}$$

is a W^*-algebra by part (iv) of Proposition 8. To see that Γ is commutative, let $L, M \in \Gamma$ and $A, B \in \mathfrak{A}$. Then

$$(L\langle A, A\rangle)B = L(\langle B, A\rangle A) = \langle B, A\rangle LA = \langle LA, A\rangle B,$$

so $L\langle A,A\rangle = \langle LA,A\rangle$, and hence

$$\langle A,A\rangle(LM-ML) = L\langle A,A\rangle M - ML\langle A,A\rangle = \langle LA,A\rangle M - M\langle LA,A\rangle = 0.$$

Therefore, $LM = ML$ since $\langle A,A\rangle A = 0$ implies $A = 0$.

Let $E \in \Gamma$ be a non-zero (self-adjoint) projection, put $\mathfrak{J} = \operatorname{Rge} E$ and let $A, B, C \in \mathfrak{A}$. If $C \in \mathfrak{J}$,

$$\langle A,B\rangle C = \langle A,B\rangle EC = E(\langle A,B\rangle C) \in \mathfrak{J}$$

and if $B \in \mathfrak{J}$,

$$\langle A,B\rangle C = \langle EB,A\rangle^* C = (E\langle B,A\rangle)^* C = \langle A,B\rangle EC = E(\langle A,B\rangle C) \in \mathfrak{J}$$

by part (iv) of Proposition 8. Hence $\mathfrak{J}$ is a J^*-ideal in $\mathfrak{A}$ and clearly $\mathfrak{J}$ contains a non-zero partial isometry by Proposition 4. Now if $E, F \in \Gamma$ are projections with $EF = 0$, then $\operatorname{Rge} E$ and $\operatorname{Rge} F$ are orthogonal sets of operators; for if $A \in \operatorname{Rge} E$ and $B \in \operatorname{Rge} F$, then $\langle A,B\rangle B \in \operatorname{Rge} E \cap \operatorname{Rge} F$, so A and B are orthogonal by [4, p. 18]. Hence there can exist at most $\mathrm{r}(\mathfrak{A})$ mutually orthogonal non-zero projections in Γ by Corollary 5. Let $E_1, \cdots, E_n$ be a maximal set of such projections and let $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$ be their corresponding ranges. Obviously, $E_1 + \cdots + E_n = I$, so (4) holds.

To show that each $\mathfrak{J}_k$ is simple, let $\mathfrak{J}$ be a J^*-ideal in $\mathfrak{J}_k$ and let E be the projection of $\mathfrak{A}$ onto $\mathfrak{J}$. Given $A, B \in \mathfrak{A}$, write $A = A_1 + A_2$ and $B = B_1 + B_2$, where $A_1, B_1 \in \operatorname{Rge} E_k$ and $A_2, B_2 \in \operatorname{Rge}(I - E_k)$. Then $\langle A,B\rangle EC = \langle A_1,B_1\rangle EC \in \mathfrak{J}$ for all $C \in \mathfrak{A}$, so $\langle A,B\rangle E = E\langle A,B\rangle E$. Hence $E \in \Gamma$ by part (iv) of Proposition 8. Since E and $E_k - E$ are mutually orthogonal projections with sum E_k, it follows that $E = 0$ or $E = E_k$ i.e., $\mathfrak{J} = \{0\}$ or $\mathfrak{J} = \mathfrak{J}_k$.

Suppose $\mathfrak{J}_1, \cdots, \mathfrak{J}_m$ are mutually orthogonal non-zero simple J^*-ideals in $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{J}_1 + \cdots + \mathfrak{J}_m$. Let A be a non-zero element of a given $\mathfrak{J}_k$ and write $A = A_1 + \cdots + A_m$, where $A_l \in \mathfrak{J}_l$ for $l = 1, \cdots, m$. There is an l such that $AA_l^*A \neq 0$, so $\mathfrak{J}_k \cap \mathfrak{J}_l$ is a non-zero J^*-ideal in both $\mathfrak{J}_k$ and $\mathfrak{J}_l$, and therefore, $\mathfrak{J}_k = \mathfrak{J}_k \cap \mathfrak{J}_l = \mathfrak{J}_l$. Conversely, by symmetry, every $\mathfrak{J}_l$ is an $\mathfrak{J}_k$. Thus the two sets of ideals agree.

To prove (6), note that by the Riesz representation theorem, there is an $L \in \mathscr{L}(\mathfrak{A})$ with $f(A,B) = (A, LB)$ for $A, B \in \mathfrak{A}$. Given $A, B, C, D \in \mathfrak{A}$, by hypothesis and part (iv) of Proposition 8,

$$\begin{aligned}(D, L(\langle A,B\rangle C)) &= f(D, \langle A,B\rangle C) = f(\langle B,A\rangle D, C)\\ &= (\langle B,A\rangle D, LC) = (D, \langle A,B\rangle LC),\end{aligned}$$

so $L \in \Gamma$. Now if $E \in \Gamma$ is a projection then $E_k E$ and $E_k(I - E)$ are mutually orthogonal projections in Γ whose sum is E_k, so $E_k E = 0$ or $E_k(I - E) = 0$ for each $k = 1, \cdots, n$. Hence E is in the span of $E_1, \cdots, E_n$. Therefore, by the spectral theorem, L is in the span of $E_1, \cdots, E_n$ and (6) follows.

To prove Corollary 11, note that by [4, theorem 6], the open unit ball $\mathcal{D}$ of $\mathfrak{J}_k$ is a bounded symmetric domain which is not holomorphically equivalent to a product of balls. Since each of the Cartan domains is holomorphically equivalent to a ball by [21, p. 286], it follows from a celebrated theorem of E. Cartan [1] that $\mathcal{D}$ is holomorphically equivalent to a Cartan domain and this domain is not exceptional by [14]. Hence $\mathcal{D}$ is holomorphically equivalent to the open unit ball of one of the Cartan factors $\mathfrak{A}$ of types I–IV and therefore $\mathfrak{J}_k$ is J^*-isomorphic to $\mathfrak{A}$ by [4, corol. 4].

Clearly Corollary 11 improves theorem 7 of [4]. (To correct the proof given there, note that by induction one can assume that $\mathfrak{A}$ is not J^*-isomorphic to a product of two J^*-algebras. Hence $\mathfrak{A}$ is J^*-isomorphic to one of the Cartan factors of type I–IV by our argument for Corollary 11.)

5. The algebraic metric

Let $\mathcal{D}$ be a bounded domain in a normed linear space X. We call an upper semicontinuous function $\alpha: \mathcal{D} \times X \to R$ an *infinitesimal Hermitian metric* on $\mathcal{D}$ if $v \to \alpha(x, v)$ is a Hilbert norm on X for each $x \in \mathcal{D}$. We say that α is *invariant* if

$$\alpha(h(x), Dh(x)v) = \alpha(x, v)$$

for all biholomorphic functions $h: \mathcal{D} \to \mathcal{D}$ and all $x \in \mathcal{D}$, $v \in X$. It follows from [5, lemma 1] that the integrated form ρ of α is a pseudometric on $\mathcal{D}$ in which every biholomorphic mapping of $\mathcal{D}$ is an isometry. For example, if X is finite dimensional, the Bergman metric β for $\mathcal{D}$ is an invariant infinitesimal Hermitian metric on $\mathcal{D}$ by [2, theorem 5.2].

To give another example, let $\mathfrak{A}$ be a J^*-algebra of finite rank, let $\|\cdot\|_2$ be the Hilbert norm given by the algebraic inner product of $\mathfrak{A}$, and define

$$\alpha(B, A) = \|(I - BB^*)^{-\frac{1}{2}} A (I - B^*B)^{-\frac{1}{2}}\|_2$$

for $B \in \mathfrak{A}_0$ and $A \in \mathfrak{A}$. Clearly α is an infinitesimal Hermitian metric on $\mathfrak{A}_0$. We shall see shortly that α is invariant. We call α the *infinitesimal algebraic metric for* $\mathfrak{A}_0$ and we call its integrated form ρ the *algebraic metric for* $\mathfrak{A}_0$.

Let $\mathfrak{A}$ be any J^*-algebra. It follows from [5, prob. 8] that the infinitesimal

CRF-metric α_c on $\mathfrak{A}_0$ is given by

$$\alpha_c(B,A) = \|(I - BB^*)^{-\frac{1}{2}} A (I - B^*B)^{-\frac{1}{2}}\|$$

for $B \in \mathfrak{A}_0$ and $A \in \mathfrak{A}$ and it follows from [5, prob. 6] that the integrated form ρ_c of α_c is given by

$$\rho_c(B,C) = \tanh^{-1}\|T_{-B}(C)\|$$

for $B, C \in \mathfrak{A}_0$. Clearly by part (ii) of Proposition 8, if $\mathfrak{A}$ has finite rank, then

$$\alpha_c \leqq \alpha \leqq \sqrt{\mathrm{r}(\mathfrak{A})}\,\alpha_c \tag{7}$$

and hence

$$\rho_c \leqq \rho \leqq \sqrt{\mathrm{r}(\mathfrak{A})}\,\rho_c. \tag{8}$$

Thus the ρ and norm topologies for $\mathfrak{A}_0$ are equivalent.

THEOREM 12 (Schwarz–Pick inequality). *Let α and α_c denote the infinitesimal algebraic and* CRF-*metrics, and let ρ and ρ_c be their integrated forms, respectively. Suppose $\mathfrak{B}$ has finite rank r and let $h: \mathfrak{A}_0 \to \mathfrak{B}_0$ be a holomorphic function. Then*

$$\alpha(h(B), Dh(B)A) \leqq \sqrt{r}\,\alpha_c(B,A),$$

$$\rho(h(B), h(C)) \leqq \sqrt{r}\,\rho_c(B,C)$$

for all $A \in \mathfrak{A}$ and $B, C \in \mathfrak{A}_0$. If h is biholomorphic, then $\mathfrak{A}$ also has rank r and

$$\alpha(h(B), Dh(B)A) = \alpha(B,A),$$

$$\rho(h(B), h(C)) = \rho(B,C)$$

for all $A \in \mathfrak{A}$ and $B, C \in \mathfrak{A}_0$.

PROOF. The first two inequalities follow immediately from (7) and (8) and the corresponding Schwarz–Pick inequalities for the CRF case [5, prob. 3]. Alternately, let $B \in \mathfrak{A}_0$ define $h_B = T_{h(B)}^{-1} \circ h \circ T_B$ and put $L = Dh_B(0)$. By the chain rule,

$$DT_{h(B)}^{-1}(h(B))[Dh(B)A] = L(DT_B^{-1}(B)A).$$

By the Cauchy estimates, $\|L\| \leqq 1$, so $\|LA\|_2^2 \leqq \mathrm{r}\|A\|^2$ for $A \in \mathfrak{A}$ by part (ii) of Proposition 8. If h is biholomorphic, then L is a J^*-isomorphism by theorems 1 and 4 of [4], so $\|LA\|_2^2 = \|A\|_2^2$ for all $A \in \mathfrak{A}$ by part (iii) of Proposition 8. Thus the theorem follows from [5, lemma 1].

Our next result shows that the infinitesimal algebraic metric is the unique invariant infinitesimal Hermitian metric on $\mathfrak{A}_0$ for which the constant in the Schwarz–Pick inequality is a minimum.

THEOREM 13. *If there is an invariant infinitesimal Hermitian metric α on $\mathfrak{A}_0$ and a number M satisfying*

$$\alpha(h(B), Dh(B)A) \leqq M\alpha(B, A) \tag{9}$$

for all holomorphic functions $h: \mathfrak{A}_0 \to \mathfrak{A}_0$ and all $A \in \mathfrak{A}$ and $B \in \mathfrak{A}_0$, then $M^2 \geqq \mathrm{r}(\mathfrak{A})$. If $M^2 = \mathrm{r}(\mathfrak{A})$ or if $\mathfrak{A}$ is simple, then α is a positive multiple of the infinitesimal algebraic metric for $\mathfrak{A}_0$. Conversely, if α is a positive multiple of the infinitesimal algebraic metric for $\mathfrak{A}_0$, then (9) holds with $M^2 = \mathrm{r}(\mathfrak{A})$.

COROLLARY 14. *The infinitesimal* CRF-*metric for $\mathfrak{A}_0$ is Hermitian if and only if $\mathfrak{A}$ is J^*-isomorphic to a Hilbert space.*

In particular, since the infinitesimal Bergman metric β is not a multiple of the infinitesimal algebraic metric for most bounded symmetric domains, inequality (9) does not hold with $\alpha = \beta$ and $M^2 = \mathrm{r}(\mathfrak{A})$, contrary to widely quoted assertions of Look [13, theorem B] and Korányi [11].

EXAMPLE. Let $\mathfrak{A}$ be the J^*-algebra of all matrices

$$A = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & z_3 \end{bmatrix},$$

where $z_1, z_2, z_3 \in \mathbf{C}$. Then $\mathrm{r}(\mathfrak{A}) = 2$ and

$$\mathfrak{A}_0 = \{A \in \mathfrak{A}: |z_1| < 1, |z_2|^2 + |z_3|^2 < 1\}.$$

Let α and β be the infinitesimal algebraic and Bergman metrics for $\mathfrak{A}_0$, respectively, and define a linear map $L: \mathfrak{A}_0 \to \mathfrak{A}_0$ by

$$L\begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \end{bmatrix}.$$

Then

$$\alpha(0, A)^2 = |z_1|^2 + |z_2|^2 + |z_3|^2,$$

$$\beta(0, A)^2 = 2|z_1|^2 + 3(|z_2|^2 + |z_3|^2)$$

for $A \in \mathfrak{A}_0$, and $\beta(0, L(A))^2 = (\frac{5}{2})\beta(0, A)^2$ when

$$A = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{A}_0.$$

But $\frac{5}{2} > \mathrm{r}(\mathfrak{A})$, so one obtains a better Schwarz–Pick inequality with α than with β.

PROOF OF THEOREM 13 AND COROLLARY 14. The last part of Theorem 13 follows from Theorem 12 and (7). Suppose $\mathrm{r}(\mathfrak{A}) < \infty$ and let $r_1, \cdots, r_n$ be the respective ranks of the simple J^*-ideals $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$ of Theorem 10. By Corollary 5, for each k, there exists a set of r_k mutually orthogonal minimal partial isometries in $\mathfrak{J}_k$. Let V be the sum of these partial isometries over all k. Let f be the inner product such that $\alpha(0, A)^2 = f(A, A)$ for $A \in \mathfrak{A}$, and note that f satisfies (5) by hypothesis and the remark at the end of the proof of Proposition 8. Choose p so that $c_p = \min\{c_1, \cdots, c_n\}$, where $c_1, \cdots, c_n$ are as in (6), choose a minimal non-zero partial isometry W in $\mathfrak{J}_p$, and define $L(A) = \ell_W(A)V$ for $A \in \mathfrak{A}$. Then $L(\mathfrak{A}_0) \subseteq \mathfrak{A}_0$, so $f(LW, LW) \leqq M^2 f(W, W)$ by hypothesis. Since $f(W, W) = c_p$, $LW = V$ and $f(V, V) = \Sigma_1^n c_k r_k$, we have

$$M^2 \geqq \sum_1^n r_k \left(\frac{c_k}{c_p}\right) \geqq \sum_1^n r_k = \mathrm{r}(\mathfrak{A}).$$

If $M^2 = \mathrm{r}(\mathfrak{A})$, then $c_k = c_p$ for $k = 1, \cdots, n$, so f/c_p is the algebraic inner product for $\mathfrak{A}$ by (6) and part (v) of Proposition 8. This obviously holds also when $\mathfrak{A}$ is simple. Since α is invariant, it follows that α/c_p is the infinitesimal algebraic metric for $\mathfrak{A}$.

To prove that $\mathrm{r}(\mathfrak{A}) < \infty$, let f be as before and observe that (5) still holds. By the upper semi-continuity of α and the open mapping theorem, there exist positive numbers m and M such that $m\| A \| \leqq \alpha(0, A) \leqq M\| A \|$ for all $A \in \mathfrak{A}$. Given $A \in \mathfrak{A}$, if $\sigma(A^*A)$ has n distinct elements, there exist continuous real functions $\varphi_1, \cdots, \varphi_n$ defined on the real line such that $\varphi_k(A^*A) \neq 0$ and $\varphi_k \varphi_j = 0$ for $k \neq j$. Put $A_k = A\varphi_k(A^*A)$ and note that $A_1, \cdots, A_n$ are orthogonal elements of $\mathfrak{A}$. Clearly, $A_k = \varphi_k(\langle A, A \rangle)A$ and $\varphi_k(\langle A, A \rangle)$ is self-adjoint with respect to f since $\langle A, A \rangle$ is, so

$$f(A_k, A_j) = f(A, A\varphi_k(A^*A)\varphi_j(A^*A)) = 0$$

for $k \neq j$. Without loss of generality we may assume that $\| A_k \| = 1$ for each k. Then

$$m^2 n \leqq f(A_1, A_1) + \cdots + f(A_n, A_n) = \alpha(0, A_1 + \cdots + A_n)^2 \leqq M^2$$

Hence $\mathrm{r}(\mathfrak{A}) \leqq (M/m)^2$.

To deduce Corollary 14, suppose α_c is Hermitian. Then α_c is invariant and (9) holds with $\alpha = \alpha_c$ and $M = 1$ by [5, prop. 3]. Hence $\mathrm{r}(\mathfrak{A}) = 1$ by Theorem 13 and therefore $\mathfrak{A}$ is J^*-isomorphic to a Hilbert space by Proposition 7. The converse is immediate from (7).

It is easy to see from the arguments for the previous two theorems that α is an invariant infinitesimal Hermitian metric on $\mathfrak{A}_0$ if and only if $\mathfrak{A}$ has finite rank and

$$\alpha(B, A) = \|(1 - BB^*)^{-\frac{1}{2}} A (I - B^*B)^{-\frac{1}{2}}\|_1$$

for all $B \in \mathfrak{A}_0$ and $A \in \mathfrak{A}$, where $\|\cdot\|_1$ is the norm on $\mathfrak{A}$ induced by a function f satisfying (6) where all the c_k's are positive and where two c_k's are equal when the corresponding $\mathfrak{J}_k$'s are J^*-isomorphic.

From now on, suppose that $\dim \mathfrak{A} < \infty$. (Note that by Corollary 11, the open unit balls of the finite dimensional J^*-algebras are just the classical bounded symmetric domains, i.e., finite products of Cartan domains of type I–IV.)

PROPOSITION 15. *Let β be the infinitesimal Bergman metric for $\mathfrak{A}_0$, let d be its integrated form, and let ℓ be the rank of $\mathfrak{A}_0$ as a Hermitian symmetric space. Let $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$ be the simple ideals in the decomposition* (4) *of $\mathfrak{A}$, and define. $M_k = n + m$, $n + 1, 2(n-1)$ or n when $\mathfrak{J}_k$ is J^*-isomorphic to* $\mathrm{I}(m, n)$, $\mathrm{II}(n)$, $\mathrm{III}(n)$ *or* $\mathrm{IV}(n)$ with $n > 2$, *respectively. Then a given one of the inequalities*

$$\beta(h(B), Dh(B)A) \leqq \sqrt{\ell}\, \beta(B, A), \tag{10}$$

$$d(h(B), h(C)) \leqq \sqrt{\ell}\, d(B, C) \tag{11}$$

holds for all holomorphic functions $h: \mathfrak{A}_0 \to \mathfrak{A}_0$ and all $A \in \mathfrak{A}$ and $B, C \in \mathfrak{A}_0$ if and only if the values of M_k agree for all k. Moreover, $\ell = \mathrm{r}(\mathfrak{A})$ and

$$\beta(0, A)^2 = 2\operatorname{tr}\langle A, A\rangle \tag{12}$$

for all $A \in \mathfrak{A}$.

Define $\tanh^{-1} B = \Sigma_0^\infty B(B^*B)^n/(2n+1)$ for $B \in \mathfrak{A}_0$. Note that if $B = \Sigma_1^n b_k V_k$ is the decomposition of Proposition 4, then

$$\tanh^{-1} B = \sum_{k=1}^{n} (\tanh^{-1} b_k) V_k. \tag{13}$$

PROPOSITION 16. *Let α be an invariant infinitesimal Hermitian metric on $\mathfrak{A}_0$ and let ρ be its integrated form. Then α is Kählerian, ρ is a C^1-metric with derivative α and*

(14) $$\rho(B, C) = \alpha(0, \tanh^{-1} T_{-B}(C))$$

for all $B, C \in \mathfrak{A}_0$. *If* α *is the infinitesimal algebraic metric, then*

(15) $$\rho(B, C)^2 = \sum_{k=1}^{n} (\tanh^{-1} d_k)^2,$$

where $d_1, \cdots, d_n$ *are the singular values of* $T_{-B}(C)$.

Proof of Propositions 15 and 16. By Corollary 11, the numbers M_k are well defined and $\ell = \mathrm{r}(\mathfrak{A})$ by [6, p. 354], Corollary 6 and the discussion preceding Proposition 4. Let α be the infinitesimal algebraic metric for $\mathfrak{A}_0$, and let $\alpha_1, \cdots, \alpha_n$ and $\beta_1, \cdots, \beta_n$ be the infinitesimal algebraic and Bergman metrics for the open unit balls of $\mathfrak{I}_1, \cdots, \mathfrak{I}_n$, respectively. Comparing the expressions for the algebraic inner product computed before Proposition 9 with those for β given in [16], we see that for each k,

(16) $$\alpha_k^2 = p_k \beta_k^2,$$

where $p_k = 1/M_k$. Hence by part (v) of Proposition 8,

$$\alpha(0, A)^2 = \sum_{k=1}^{n} p_k \beta_k(0, A_k)^2$$

for all $A \in \mathfrak{A}$. (Here and in the sequel, we write A_k and B_k for the k-th coordinate of the respective decompositions of A and B given by (4).) On the other hand,

(17) $$\beta(0, A)^2 = \sum_{k=1}^{n} \beta_k(0, A_k)^2$$

for all $A \in \mathfrak{A}$ by [2, theorem 5.4]. Hence β is a positive multiple c of α if and only if $M_k = c^2$ for all k. This together with Theorem 13 proves Proposition 15 for (10). Note that (10) and (11) are equivalent by Proposition 16 and [5, prob. 7b].

To prove (12), define $f(A, B) = 2\mathrm{tr}\langle A, B\rangle$ for $A, B \in \mathfrak{A}$. If $L: \mathfrak{A} \to \mathfrak{A}$ is a J^*-isomorphism, then $f(LA, LB) = f(A, B)$ since $L\langle A, B\rangle L^{-1} = \langle LA, LB\rangle$, so f satisfies (5) by the remark at the end of the proof of Proposition 8. Hence by Theorem 10 and (17), it suffices to verify that (12) holds for some non-zero element V of $\mathfrak{A}$ when $\mathfrak{A}$ is a Cartan factor. This verification is trivial when $\mathfrak{A}$ is of types II, IV and III(n) with n even since we may take V to be unitary, and in the remaining cases we may take V to be a non-zero minimal element of $\mathfrak{A}$.

Now let α be an invariant infinitesimal Hermitian metric on $\mathfrak{A}_0$. Then as in the proof of Theorem 13, there are positive numbers $c_1, \cdots, c_n$ with

$$\alpha(0, A)^2 = \sum_{k=1}^{n} c_k \alpha_k (0, A_k)^2 \tag{18}$$

for all $A \in \mathfrak{A}$. Let $K_1, \cdots, K_n$ be the Bergman kernel functions for the open unit balls of $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$. Since for each k, $\log K_k$ gives rise to a Kähler potential function for β_k, it follows that $\log \Phi$ gives rise in the same way to a Kähler potential function for α, where

$$\Phi(A, B) = K_1(A_1, B_1)^{q_1} \cdots K_n(A_n, B_n)^{q_n}$$

and $q_k = c_k p_k$ for all k. Hence α is Kählerian.

To prove (14), let $E_1, \cdots, E_n$ be the projections onto $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$ and let $\rho_1, \cdots, \rho_n$ be the algebraic metrics on the open unit balls of $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$, respectively. Given $B \in \mathfrak{A}_0$, let γ be a curve in $\mathfrak{A}_0$ with piecewise continuous derivative and suppose $\gamma(0) = 0$ and $\gamma(1) = B$. Put $\gamma_k = E_k \circ \gamma$ for each k. Then

$$\begin{aligned} &[I - \gamma(t)\gamma(t)^*]^{-\frac{1}{2}} \gamma'(t) [1 - \gamma(t)^* \gamma(t)]^{-\frac{1}{2}} \\ &= \sum_{k=1}^{n} [I - \gamma_k(t)\gamma_k(t)^*]^{-\frac{1}{2}} \gamma_k'(t) [I - \gamma_k(t)^* \gamma_k(t)]^{-\frac{1}{2}}, \end{aligned}$$

so

$$\alpha(\gamma(t), \gamma'(t))^2 = \sum_{k=1}^{n} c_k \alpha_k (\gamma_k(t), \gamma_k'(t))^2$$

for all $0 \leqq t \leqq 1$ by (18). Hence by Minkowski's inequality,

$$L_\alpha(\gamma)^2 \geqq \sum_{k=1}^{n} c_k L_{\alpha_k}(\gamma_k)^2 \geqq \sum_{k=1}^{n} c_k \rho_k (0, B_k)^2.$$

Now for each k, let $B_k = \Sigma_1^{n_k} b_l V_l$ be the decomposition of Proposition 4 and let γ_k be the curve $\gamma_k(t) = \Sigma_1^{n_k} b_l(t) V_l$, where $b_l(t) = \tanh(t \tanh^{-1} b_l)$ for $l = 1, \cdots, n_k$. Then γ_k is a curve in the open unit ball $\mathcal{D}$ of $\mathfrak{J}_k$ and

$$[I - \gamma_k(t)\gamma_k(t)^*]^{-\frac{1}{2}} \gamma_k'(t) [1 - \gamma_k(t)^* \gamma_k(t)]^{-\frac{1}{2}} = \sum_{l=1}^{n_k} (\tanh^{-1} b_l) V_l$$

for $0 \leqq t \leqq 1$. By (16) and [16, p. 19–20], γ_k is a curve in $\mathfrak{D}$ connecting 0 to B_k of shortest length with respect to α_k, so

$$\rho_k(0, B_k)^2 = L_{\alpha_k}(\gamma_k)^2 = \sum_{l=1}^{n_k} (\tanh^{-1} b_l)^2 = \alpha_k(0, \tanh^{-1} B_k)^2.$$

Let $\gamma = \gamma_1 + \cdots + \gamma_n$. Clearly γ is a curve in $\mathfrak{A}_0$ with $\gamma(0) = 0$ and $\gamma(1) = B$. Applying (18) and the fact that $\alpha_k(\gamma_k(t), \gamma_k'(t))$ is constant in t, we have

$$L_\alpha(\gamma)^2 = \sum_{k=1}^{n} c_k L_{\alpha_k}(\gamma_k)^2 = \sum_{k=1}^{n} c_k \alpha_k(0, \tanh^{-1} B_k)^2 = \alpha(0, \tanh^{-1} B)^2$$

since $(\tanh^{-1} B)_k = \tanh^{-1} B_k$ for all k. Thus γ is a curve in $\mathfrak{A}_0$ connecting 0 to B of shortest length with respect to α and $\rho(0, B) = \alpha(0, \tanh^{-1} B)$. Hence (14) follows since biholomorphic mappings of $\mathfrak{A}_0$ are ρ-isometries and (15) then follows from (13).

To show that ρ is a C^1-metric with derivative α (see [5]), define a norm $\| \quad \|_1$ on $\mathfrak{A}$ by $\| A \|_1 = \alpha(0, A)$ and note that there is a number M satisfying $\| A \|_1 \leqq M \| A \|$ for all $A \in \mathfrak{A}$. Given numbers r and s with $0 < r < 1$ and $0 < s < 1 - r$, let $A, B \in \mathfrak{A}$ satisfy $\| B \| < r$ and $\| A \| < s$. Put

$$C = T_{-B}(B + A), \qquad D = (I - BB^*)^{-\frac{1}{2}} A (I - B^*B)^{-\frac{1}{2}},$$
$$R = (I - BB^*)^{-\frac{1}{2}} A (I - B^*B - B^*A)^{-1} B^*A (I - B^*B)^{-\frac{1}{2}},$$

and observe that $C = D + R$. By (14),

$$| \rho(B + A, B) - \alpha(B, A) | = | \| \tanh^{-1} C \|_1 - \| D \|_1 | \leqq \| \tanh^{-1} C - D \|_1.$$

Now $\| \tanh^{-1} C - C \| \leqq \tanh^{-1} \| C \| - \| C \| \leqq \| C \|^2 / (1 - \| C \|)$, and there exist numbers K_1, K_2 and p depending only on r and s such that $\| R \| \leqq K_1 \| A \|^2$, $\| C \| \leqq p < 1$ and $\| C \| \leqq K_2 \| A \|$. Hence

$$\| \tanh^{-1} C - D \| \leqq \| \tanh^{-1} C - C \| + \| R \| \leqq \left(K_1 + \frac{K_2^2}{1 - p} \right) \| A \|^2.$$

Thus there is a number Q depending inly on r and s such that

$$| \rho(B + A, B) - \alpha(B, A) | \leqq Q \| A \|^2.$$

This completes the proof.

Note that the triangle inequality for the algebraic metric (i.e., $\rho(B, C) \leqq \rho(B, O) + \rho(O, C)$) is a rather subtle inequality between singular values.

We conjecture that Proposition 9, Corollary 11 and Proposition 16 (excluding the assertion that α is Kählerian) hold without the assumption that $\dim \mathfrak{A} < \infty$ and that the representation (1) is unique up to order of terms.

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